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THE LIMITATIONS OF THE MODEL OF A CRACK ALONG THE INTERFACE OF AN INCLUSION IN AN ELASTIC MATRIX

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Abstract—The two-dimensional problem of an arc-shaped crack lying along the interface of a fixed circular rigid inclusion embedded in an elastic matrix is considered. The assumption that the inclusion is rigid provides the same mathematical behavior of an oscillatory singularity at the crack tips as if an elastic inclusion had been assumed but in a simpler mathematical problem. In contrast to previous studies which determine only the radial displacement of the crack faces, a simple closed-form formula for the entire displacement *throughout* the elastic matrix is determined for a far-field biaxial load with an infinitesimal rigid rotation. From this expression a natural decomposition of the problem is identified which allows interpenetration regions to be determined easily. It is found that most loading situations predict the elastic crack face to have a large interpenetration region. Thus if the behavior of the material near the crack is to be accurately determined it is necessary to use a model which allows the crack faces to be in contact.

INTRODUCTION

In fiber-reinforced composites, an important mechanism of damage growth is the debonding of the fiber-matrix interface. One of the first steps toward a better understanding of such damage growth is to construct and solve simplified mathematical models of cracks along circular interfaces. These solutions may then indicate which parameters may be important for damage growth.

The first mathematical models of a crack along a circular inclusion in an elastic matrix can be found in England (1966), Perlman and Sih (1967) and Toya (1973, 1974). England (1966) considered an elastic inclusion with a uniformly pressurized crack and derived an expression for the displacement of the crack faces at the interface. Perlman and Sih (1967) determined the potentials for a concentrated point force in the material which could be used as a Green function for an arbitrary loading. They also developed expressions for the stress at the interface. Toya (1973) considered a pressurized crack along a rigid inclusion and Toya (1974) assumed a fully open traction-free crack along an elastic inclusion opened by a biaxial tension at infinity. In Toya (1974), the derivatives of the Kolosov–Muskhelishvili potentials were determined and from these, formulae for the stress and the displacements along the interface were established. However, all these solutions exhibit the same unphysical interpenetration of the crack faces due to an oscillatory singularity similar to that found in linear elastic studies of Griffith interface cracks (e.g. England, 1965; Erdogan, 1965; Lowengrub and Sneddon, 1973; Willis, 1971). If these interpenetration zones are small, these solutions may be an acceptable approximation to the physical situation away from the crack tips. However, as will be seen below, only hydrostatic tension at infinity always predicts small interpenetration zones analogous to the Griffith interface crack under uniaxial tension normal to the crack faces. For the other natural far-field loadings in the elastic matrix, large interpenetration zones are predicted at the interface, thus making their predictions of the material behavior near the crack suspect. Therefore, in most cases, an acceptable model for these cracks must admit regions in which the crack faces come into contact.

For the following, it will be assumed that the inclusion is rigid and fixed. As will be seen, all the interesting mathematical behavior is retained in this simpler mathematical model. The biaxial tension problem with open traction-free crack faces is re-examined and the displacements and stresses are seen to be the superposition of the displacements and stresses of the three natural far-field loadings. For the first time, formulae for the stresses and displacements *throughout* the body are developed and their predictions will be examined. It is seen that large interpenetration zones are predicted in most loading situations.

STATEMENT AND ANALYSIS OF THE PROBLEM

Consider the two-dimensional problem of a body consisting of a fixed circular rigid inclusion embedded in an infinite elastic matrix. At the interface, it is assumed that these two differing materials are (i) perfectly bonded, across which stresses and displacements are continuous, or (ii) unbonded, across which displacements may be discontinuous. We shall call the unbonded region a crack. We shall denote by D^- and D^+ the regions occupied by the elastic matrix and the rigid inclusion respectively. We shall denote the interface between D^- and D^+ by C and denote that part of the interface along which the crack exists by A_S and that part of the interface where the rigid inclusion is bonded to the elastic matrix by A_D (see Fig. 1). We shall use the complex variable notation as found in Muskhelishvili (1954) and Gladwell (1980), and will place the origin of the complex plane at the center of the rigid inclusion. The inclusion will be assumed to occupy $|z| < 1$, the elastic matrix $|z| > 1$, and the axes will be placed such that a crack of length 2α, $0 < \alpha < \pi$, is symmetrically placed with respect to the positive real axis, i.e., $A_S = \{t = r\,e^{i\theta}: -\alpha < \theta < \alpha\}$. Let $\tau_{rr}(z)$, $\tau_{r\theta}(z)$ and $\tau_{\theta\theta}(z)$ denote the polar stresses and u and v denote the displacements parallel to the real and imaginary axes respectively. We have the following mathematical problem. Determine $(u, v, \tau_{rr}, \tau_{r\theta}, \tau_{\theta\theta})$ such that they satisfy the equations of linear elastostatic equilibrium in the absence of body forces for plane strain or plane stress conditions with boundary conditions:

$$\text{traction-free crack faces} \quad \tau_{rr} + i\tau_{r\theta} = 0 \quad \text{on} \quad A_S, \tag{1}$$

$$\text{continuity of displacements} \quad u + iv = 0 \quad \text{on} \quad A_D. \tag{2}$$

The Kolosov–Muskhelishvili potentials ϕ and ψ will automatically satisfy the equations of equilibrium and the constitutive law of linear elasticity for the elastic matrix if they are analytic in D^- and the displacements and polar stresses are expressed as

$$2\mu(u+iv) = \kappa\phi(z) - z\bar{\phi}'(\bar{z}) - \bar{\psi}(\bar{z}) \tag{3}$$

$$\Theta \equiv \tau_{rr} + \tau_{\theta\theta} = 2[\Phi(z) + \bar{\Phi}(\bar{z})] \tag{4}$$

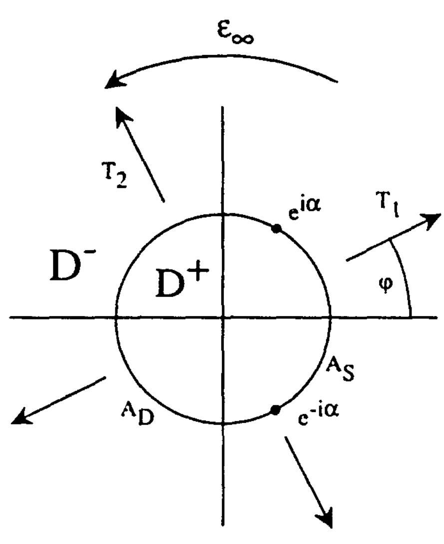

Fig. 1. The geometry of the fixed rigid inclusion D^+, the elastic matrix D^-, and the far-field loading of a biaxial tension and infinitesimal rigid rotation.

$$\Xi\, e^{2i\theta} \equiv \tau_{\theta\theta} - \tau_{rr} + 2i\tau_{r\theta} = 2\, e^{2i\theta}\,[\bar{z}\Phi'(z) + \Psi(z)] \tag{5}$$

where $\bar{\psi}(z) = \overline{\psi(\bar{z})}$, $\Phi = \phi'$, $\Psi = \psi'$, and, under plane strain conditions, $\kappa = 3-4\nu$ where ν is Poisson's ratio. For generalized plane stress conditions, $\kappa = 3-4\nu^*$ where $\nu^* = \nu/(1+\nu)$. Thus if $0 < \nu < \frac{1}{2}$ then in plane strain $1 < \kappa < 3$ and in generalized plane stress $(5/3) < \kappa < 3$.

Thus the entire problem becomes to determine analytic functions in D^- which satisfy the proper boundary conditions. These boundary conditions are of two basic types, the far-field loading conditions of a biaxial tension and infinitesimally rigid rotation and the boundary conditions of an open crack or perfect bond at the interface.

Let us consider the far-field loading conditions first. We shall require stresses and displacements to be single valued in the elastic matrix, the stresses to be bonded near infinity, and, since the body is in equilibrium, the total force applied to the elastic matrix across the interface to be zero. Thus $\Phi(z)$ and $\Psi(z)$ must be bounded and each of the form $k+o(z)$ as $z \to \infty$ (see Gladwell, 1980). A biaxial tension at infinity with principal stresses T_1 and T_2 where T_1 is at an angle φ with the positive real axis (see Fig. 1) requires

$$\Theta(\infty) = \tau_{rr}(\infty) + \tau_{\theta\theta}(\infty) = T_1 + T_2$$

and

$$\Xi(\infty) = e^{-2i\theta}\,[\tau_{\theta\theta}(\infty) - \tau_{rr}(\infty) + 2i\tau_{r\theta}(\infty)] = (T_2 - T_1)\, e^{-2i\varphi}. \tag{6}$$

From (4) then $\operatorname{Re}\Phi(\infty) = \frac{1}{4}(T_1 + T_2)$ and thus

$$\Phi(z) = \tfrac{1}{4}(T_1 + T_2) + iC_0 + o(z^{-1}) \tag{7}$$

as $z \to \infty$, where C_0 is real and undetermined.

Furthermore, from (5)–(7) then

$$\Psi(z) = \tfrac{1}{2}(T_2 - T_1)\, e^{-2i\varphi} + o(z^{-1}) \quad \text{as} \quad z \to \infty. \tag{8}$$

By considering the potentials $\Phi(z) = iC_0$ and $\Psi(z) = 0$ it can be shown that the displacement corresponding to these potentials is an infinitesimally rigid rotation about zero (or equivalently infinity), for which the stresses are zero throughout the complex plane. Thus if we define ε_∞ to be the prescribed infinitesimally rigid rotation about infinity then

$$C_0 = \frac{2\mu\varepsilon_\infty}{1+\kappa}.$$

Thus the far-field loading conditions of biaxial tension and a prescribed infinitesimally rigid rotation ε_∞ about ∞ determine the asymptotics of $\Phi(z)$ and $\Psi(z)$ as $z \to \infty$.

We shall now consider the boundary conditions at the interface. In an effort to simplify the boundary conditions there, we will extend the potential $\Phi(z)$ which is defined only for the elastic matrix D^- to the complementary domain D^+ by

$$\Phi(z) = -\bar{\Phi}\left(\frac{1}{z}\right) + \frac{1}{z}\bar{\Phi}'\left(\frac{1}{z}\right) + \frac{1}{z^2}\bar{\Psi}\left(\frac{1}{z}\right). \tag{9}$$

Equation (9) allows us to express $\Psi(z)$ in terms of $\Phi(z)$ for $z \in D^-$ and $\Phi(w)$ where $w = (1/z) \in D^+$ as

$$\Psi(z) = z^{-2}\Phi(z) + z^{-2}\bar{\Phi}\left(\frac{1}{z}\right) - z^{-1}\Phi'(z) \quad \text{for} \quad z \in D^-. \tag{10}$$

Note that if $f(z)$ is analytic for $|z| > 1$ then $\bar{f}(1/z)$ is analytic for $0 < |z| < 1$. Since Φ and

Ψ are analytic in D^- then (9) extends Φ to be analytic in D^+ except possibly at $z = 0$. At $z = 0$ we have from (7)–(9) that

$$\Phi(z) = \tfrac{1}{2}(T_2 - T_1)\,e^{2i\varphi} z^{-2} + O(1) \quad \text{as} \quad z \to 0. \tag{11}$$

From (4), (5), (7), and $e^{-2i\theta} = \bar{z}z^{-1}$, one can obtain

$$\tau_{rr} + i\tau_{r\theta} = \Phi(z) - \Phi(1/\bar{z}) + \bar{z}(\bar{z} - z^{-1})\bar{\Psi}(\bar{z}), \quad z \in D^-. \tag{12}$$

Upon differentiating (3) and proceeding in a similar manner, one obtains

$$2\mu(u' + iv') = \kappa\Phi(z) + \Phi(1/\bar{z}) - \bar{z}(\bar{z} - z^{-1})\bar{\Psi}(\bar{z}), \quad z \in D^-. \tag{13}$$

We shall assume Φ continuous on $C = A_D \cup A_S$, as $z \to t \in C$ from either D^+ or D^- except at possibly a finite number of points c_k of C near which

$$|\Phi(z)| < A|z - c_k|^{-\alpha}, \quad 0 \leqslant \alpha < 1. \tag{14}$$

In addition it will be assumed that

$$\lim_{r \to 1} (1 - r)\Phi'(r\,e^{i\theta}) = 0 \tag{15}$$

for all values of θ, except possibly at the points c_k. It can be shown that these conditions lead to

$$\lim_{r \to 1} (\bar{z} - z^{-1})\Psi(z) = 0, \quad z = r\,e^{i\theta}. \tag{16}$$

We shall define the following notation:

$$\text{let } \Phi^-(t) \equiv \lim_{z \to t} \Phi(z),$$

where $t = e^{i\theta}$ and $z \in D^-$. $\Phi^+(t)$ will represent the complementary limit at $z \to t^+$, i.e., $z \in D^+$ and z approaches $t \in C$. Then (12) and (13) allow the boundary conditions (1) and (2) to be expressed in terms of the potential Φ as

$$\Phi^-(t) - \Phi^+(t) = 0 \quad (t \in A_S) \tag{17}$$

and

$$\kappa\Phi^-(t) + \Phi^+(t) = 0 \quad (t \in A_D). \tag{18}$$

We shall now briefly consider the problem of a perfectly bonded inclusion and identify the natural loading conditions at infinity. We shall then look at the model which includes a crack along the inclusion and assumes the crack faces are fully open and traction-free.

Problem 1

We shall assume that the interface is entirely perfectly bonded, i.e., $C = A_D$. The boundary condition of a perfect bond at the interface of the elastic matrix and fixed rigid inclusion can be written in terms of the unknown potential Φ as

$$\kappa\Phi^-(t) + \Phi^+(t) = 0 \quad \text{for} \quad t \in C. \tag{19}$$

By inspection, it may be seen that

$$\Phi(z) = \tfrac{1}{4}(T_1 + T_2) + \frac{2i\mu\varepsilon_\infty}{1 + \kappa} - \frac{1}{2\kappa}(T_2 - T_1)\,e^{2i\varphi} z^{-2} \quad \text{for} \quad z \in D$$

and

$$\Phi(z) = -\kappa\left[\tfrac{1}{4}(T_1+T_2)+\frac{2i\mu\varepsilon_\infty}{1+\kappa}\right]+\tfrac{1}{2}(T_2-T_1)\,e^{2i\varphi}z^{-2} \quad \text{for} \quad z\in D^+ \tag{20}$$

satisfies (19), (7), and (11), and are analytic for $z\in D^-$ and $z\in D^+-\{0\}$.

The displacement field throughout the elastic matrix can now be constructed and it is found that

$$2\mu(u+iv) = \tfrac{1}{4}(T_1+T_2)(\kappa-1)(z-\bar{z}^{-1})+2i\mu\varepsilon_\infty(z-\bar{z}^{-1})$$
$$+\tfrac{1}{2}(T_2-T_1)\left[e^{2i\varphi}(z^{-1}-\bar{z})+\frac{1}{\kappa}e^{-2i\varphi}\bar{z}^{-2}(z-\bar{z}^{-1})\right] \quad \text{for} \quad z\in D^-. \tag{21}$$

It can be seen from (21) that the general loading situation considered may be conveniently expressed as the superposition of the three natural loads;

(i) hydrostatic tension or compression $T_1 = T_2 = T \neq 0$ and $\varepsilon_\infty = 0$;
(ii) pure shear $T_1 = -T_2 = S \neq 0$ and $\varepsilon_\infty = 0$; and
(iii) an infinitesimally rigid rotation $\varepsilon_\infty \neq 0$, and $T_1 = T_2 = 0$.

The displacement of the elastic matrix under a hydrostatic tension ($T > 0$) consists entirely of the outward radial displacement

$$u_r(z) = \frac{T}{4\mu}(\kappa-1)(r-r^{-1}) \quad \text{where} \quad z = r\,e^{i\theta}$$

and the radial and tangential displacements, u_r and u_θ, are related to the Cartesian displacements u and v by

$$u(z)+iv(z) = e^{i\theta}\,(u_r(z)+iu_\theta(z)). \tag{22}$$

The displacement of the elastic matrix under a hydrostatic compression ($T < 0$) similarly consists entirely of the inward radial displacement

$$u_r(z) = \frac{T}{4\mu}(\kappa-1)(r-r^{-1}).$$

Thus the deformation of the matrrix is given by

$$z+u(z)+iv(z) = e^{i\theta}\left[r+\frac{T}{4\mu}(\kappa-1)(r-r^{-1})\right]$$

and it can be shown that if $(-T/2\mu) > 1/(\kappa-1)$ then the radial distance from the origin after the deformation is not monotonically increasing for all $r > 1$. This would lead to the elastic material interpenetrating itself and the inclusion. Thus it is assumed that $(-T/2\mu) \leqslant 1/(\kappa-1)$. The breakdown of the solution at this load level may be caused by large strain levels which can no longer be accurately predicted by the linear theory of elasticity. The radial and tangential displacements corresponding to a pure shear can be written as

$$u_r(z)+iu_\theta(z) = \frac{S}{2\mu}(r-r^{-1})\left[\left(1-\frac{1}{\kappa r^2}\right)\cos(2(\theta-\varphi))-i\left(1+\frac{1}{\kappa r^2}\right)\sin(2(\theta-\varphi))\right].$$

Again, so that interpenetration is not allowed

$$\frac{|S|}{2\mu} \leqslant \frac{12\kappa}{12\kappa+(\kappa+1)^2}.$$

The displacement of the elastic matrix under an infinitesimal rigid rotation consists entirely of the tangential displacement $u_\theta(z) = \varepsilon_\infty(r-r^{-1})$ where $z = r\,e^{i\theta}$. Examples of the displacements corresponding to each of these loadings are illustrated in Fig. 2.

In a similar manner, the stress fields can be determined from (4) and (5) and can be conveniently considered as the superposition of the stresses resulting from the three natural loadings.

Problem 2

We shall assume that a crack of length 2α, $\alpha > 0$, exists along the interface with crack faces open and traction-free. The boundary conditions at the interface become

$$\kappa\Phi^-(t)+\Phi^+(t) = 0 \quad \text{for} \quad t\in A_D \quad \text{and} \quad \Phi^-(t)-\Phi^+(t) = 0 \quad \text{for} \quad t\in A_S. \tag{23}$$

Note that $\Phi^-(t)-\Phi^+(t) = 0$ requires Φ to be continuous and thus analytic across A_S and therefore we are looking for a function Φ that is analytic everywhere except at zero and along A_D and which satisfies (7) and (11). We shall rewrite Φ in the following form:

$$\Phi(z) = \Phi_0(z)+h(z) \quad \text{where} \quad h(z) \equiv \tfrac{1}{4}(T_1+T_2)+\frac{2i\mu\varepsilon_\infty}{1+\kappa}+\tfrac{1}{2}(T_2-T_1)\,e^{2i\varphi}z^{-2}. \tag{24}$$

Now $\Phi_0(z)$ will be analytic everywhere in the complex plane except along A_D and $\Phi_0(z) \to 0$ as $z \to \infty$. Furthermore, the jump condition across A_D for $\Phi(z)$ in (23) becomes for $\Phi_0(z)$

$$\kappa\Phi_0^-(t)+\Phi_0^+(t) = -(\kappa+1)h(z) \quad \text{for} \quad t\in A_D. \tag{25}$$

The solution to this Riemann–Hilbert problem may be calculated from

$$\Phi_0(z) = -(\kappa+1)\frac{\chi(z)}{2\pi i}\int_{A_D}\frac{h(t)}{\chi^+(t)}\frac{\mathrm{d}t}{t-z} \tag{26}$$

where $\chi(z)$ is the solution to the homogeneous Riemann–Hilbert problem

$$\kappa\chi^-(t)+\chi^+(t) = 0 \quad \text{for} \quad t\in A_D. \tag{27}$$

It is known from previous studies that the stresses are unbounded near the endpoints $e^{i\alpha}$ and $e^{-i\alpha}$ and since

$$\tau_{rr}(t)+i\tau_{r\theta}(t) = (\kappa+1)\Phi^-(t) \quad \text{for} \quad t\in A_D,$$

the potential Φ is also unbounded there. Thus the homogeneous solution must be chosen to be

$$\chi(z) = \left[\frac{z-e^{-i\alpha}}{z-e^{i\alpha}}\right]^{\lambda_0}\frac{1}{z-e^{-i\alpha}} \quad \text{where} \quad \lambda_0 = \frac{1}{2}-\frac{i}{2\pi}\ln(\kappa) \tag{28}$$

with the branch of w^{λ_0} determined by $-\alpha < \arg w < 2\pi-\alpha$.

If this expression for χ is substituted into (26) and simplified by the integrals (A1) and

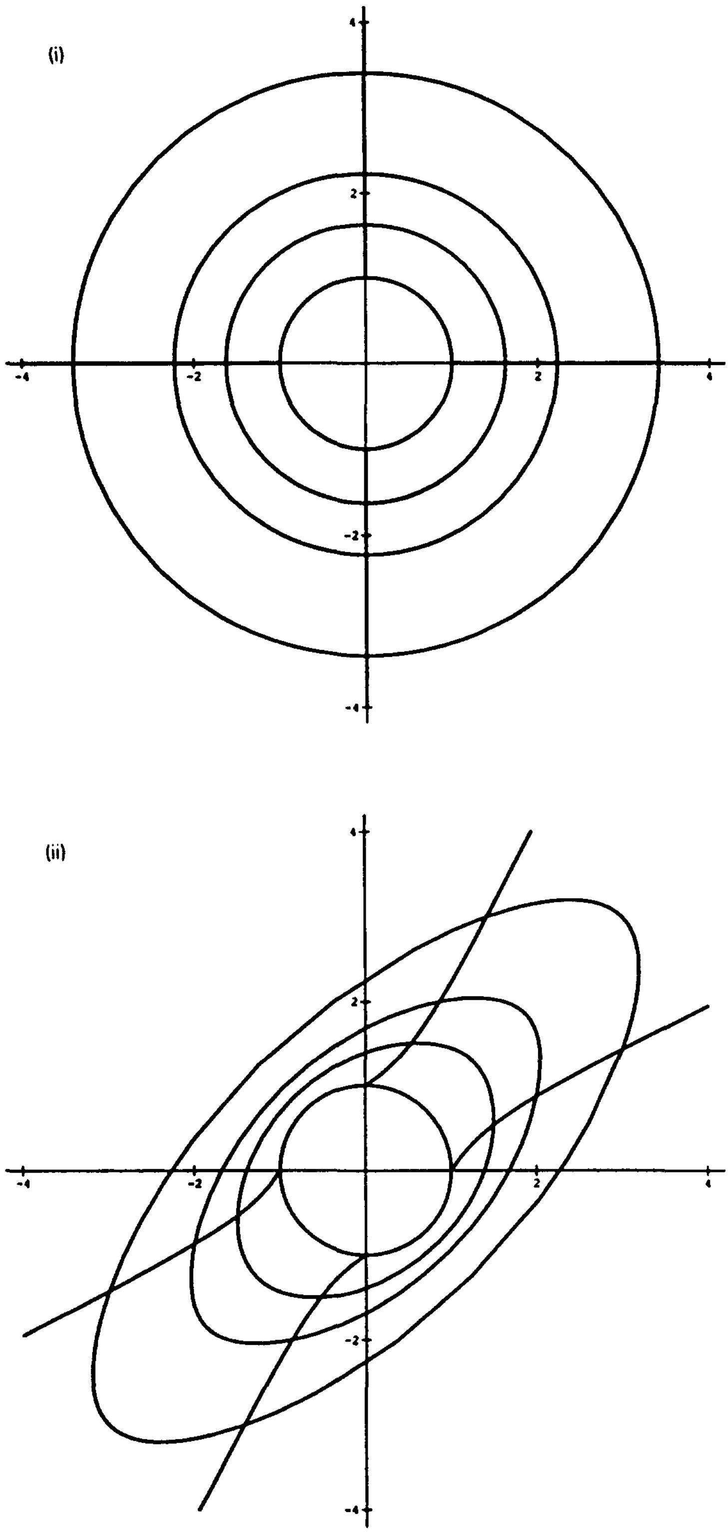

Fig. 2. The deformation of a fixed rigid inclusion perfectly bonded to an elastic matrix for the natural loads: (i) hydrostatic tension, (ii) pure shear with $\varphi = \pi/4$, and (iii) an infinitesimal rigid rotation, $\varepsilon_\infty = 0.5$. The reference configuration (R) consists of concentric circles of radii 1, 1.5, 2 and 3 and the axes.

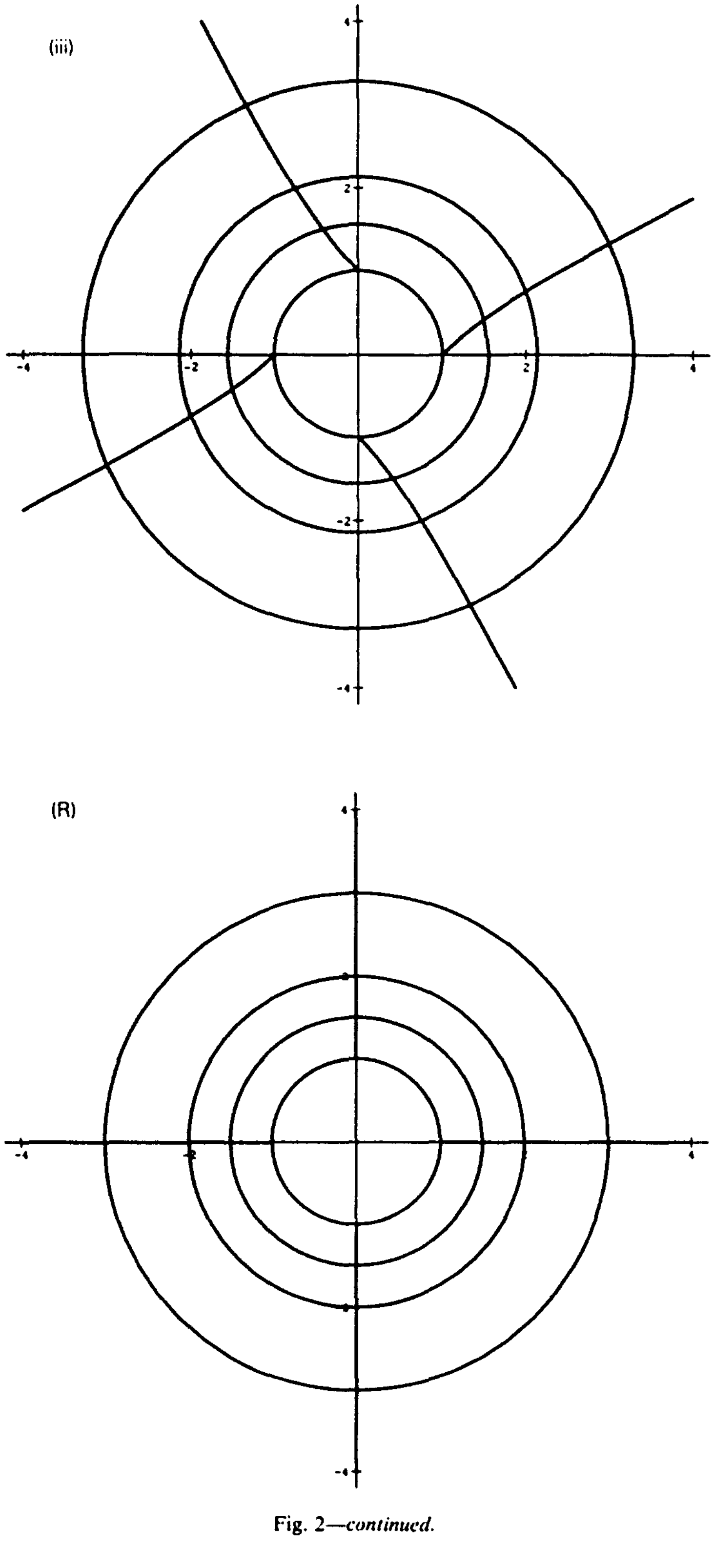

Fig. 2—*continued.*

(A2) found in the Appendix, then Φ is found to be

$$\Phi(z) = -\tfrac{1}{2}(T_2-T_1)\,e^{2i\varphi}\chi(z)\frac{1}{a_0 z^2}\left[\frac{a_1}{a_0}z-1\right]+\left[\tfrac{1}{4}(T_1+T_2)+\frac{2i\mu\varepsilon_\infty}{1+\kappa}\right]\chi(z)(z-b_2) \tag{29}$$

where a_0, a_1, and b_2 are the first few terms of the series expansion of χ about zero and infinity, respectively. Their values can be found in the Appendix.

From (29) and (10), we may now write Ψ as

$$\Psi(z) = -[z^{-1}\Phi(z)]'-\tfrac{1}{2}(T_2-T_1)\,e^{-2i\varphi}-\bar{\chi}(1/z)\frac{1}{a_0}\left[\frac{a_1}{a_0 z}-1\right]$$
$$+\left[\tfrac{1}{4}(T_1+T_2)-\frac{2i\mu\varepsilon_\infty}{1+\kappa}\right]\bar{\chi}(1/z)z^{-2}\left[\frac{1}{z}-b_2\right] \tag{30}$$

where

$$\bar{\chi}(1/z) \equiv \overline{\frac{1}{1/\bar{z}-e^{-i\alpha}}\left(\frac{1/\bar{z}-e^{-i\alpha}}{1/\bar{z}-e^{i\alpha}}\right)^{\lambda_0}} = \frac{-z\,e^{-i\alpha}}{z-e^{-i\alpha}}\left(e^{2i\alpha}\frac{z-e^{-i\alpha}}{z-e^{i\alpha}}\right)^{\lambda_0}.$$

These may be integrated using the results (A3)–(A6) in the Appendix to determine the Kolosov–Muskhelishvili potentials

$$\phi(z) = -\tfrac{1}{2}(T_2-T_1)\,e^{2i\varphi}R_1(z)\frac{1}{a_0 z}+\left[\tfrac{1}{4}(T_1+T_2)+\frac{2i\mu\varepsilon_\infty}{1+\kappa}\right]R_1(z)$$
$$\psi(z) = -z^{-1}\Phi(z)-\tfrac{1}{2}(T_2-T_1)\,e^{2i\phi}R_2(z)\frac{e^{-i\alpha}}{a_0}+\left[\tfrac{1}{4}(T_1+T_2)-\frac{2i\mu\varepsilon_\infty}{1+\kappa}\right]$$
$$\times R_2(z)\frac{e^{-i\alpha}}{z} \quad \text{for} \quad z\in D\ . \tag{31}$$

We may now determine from (3) the displacements throughout the elastic matrix

$$2\mu(u+iv) = \tfrac{1}{4}(T_1+T_2)\left[\kappa R_1(z)+(\bar{z}^{-1}-z)\overline{\chi(z)}(\bar{z}-b_2)-e^{i\alpha}\bar{z}^{-1}\overline{R_2(z)}\right]$$
$$+\frac{2i\mu\varepsilon_\infty}{1+\kappa}\left[\kappa R_1(z)-(\bar{z}^{-1}-z)\overline{\chi(z)}(\bar{z}-b_2)-e^{i\alpha}\bar{z}^{-1}\overline{R_2(z)}\right]$$
$$-\tfrac{1}{2}(T_2-T_1)\left[\kappa\,e^{2i\varphi}R_1(z)\frac{1}{a_0 z}+(\bar{z}^{-1}-z)\overline{\chi(z)}\,e^{-2i\varphi}\frac{1}{a_0\bar{z}^2}\left[\frac{a_1}{a_0}\bar{z}-1\right]\right.$$
$$\left.-e^{2i\varphi}\overline{R_2(z)}\frac{e^{i\alpha}}{a_0}\right] \quad \text{for} \quad z\in D^-. \tag{32}$$

Again it is found that the displacements may be conveniently expressed as the superposition of the three natural loads. Figure 3 shows the deformation of the elastic matrix assuming a crack along the interface from $e^{-i\pi/4}$ to $e^{i\pi/4}$ under each of these loads.

From (4), (5), (29), and (30) the stresses are determined to be

$$
\begin{aligned}
\tau_{rr}+\tau_{\theta\theta} = {} & \tfrac{1}{2}(T_1+T_2)[\chi(z)(z-b_2)+\overline{\chi(z)}(\bar{z}-b_2)]+\frac{4i\mu\varepsilon_\infty}{1+\kappa}[\chi(z)(z-b_2)-\overline{\chi(z)}(\bar{z}-b_2)] \\
& -(T_2-T_1)\left(e^{2i\varphi}\chi(z)\frac{1}{a_0z^2}\left[\frac{a_1}{a_0}z-1\right]+e^{-2i\varphi}\overline{\chi(z)}\frac{1}{a_0\bar{z}^2}\left[\frac{a_1}{a_0}\bar{z}-1\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
\tau_{\theta\theta}-\tau_{rr}+2i\tau_{r\theta} = {} & 2\,e^{2i\theta}\Bigg(\tfrac{1}{4}(T_1+T_2)\Big[(\bar{z}-z^{-1})[\chi(z)(z-b_2)]'+z^{-2}[\chi(z)(z-b_2) \\
& +\bar{\chi}(1/z)(1/z-b_2)]\Big]+\frac{2i\mu\varepsilon_\infty}{1+\kappa}\Big[(\bar{z}-z^{-1})[\chi(z)(z-b_2)]' \\
& +z^{-2}[\chi(z)(z-b_2)-\bar{\chi}(1/z)(1/z-b_2)]\Big]-\tfrac{1}{2}(T_2-T_1) \\
& \times\left[e^{2i\varphi}(\bar{z}-z^{-1})\left(\chi(z)\frac{1}{a_0z^2}\left[\frac{a_1}{a_0}z-1\right]\right)'+e^{2i\varphi}\chi(z)\frac{1}{a_0z^4}\left[\frac{a_1}{a_0}z-1\right]\right. \\
& \left.+e^{-2i\varphi}\bar{\chi}(1/z)\frac{1}{a_0z^2}\left[\frac{a_1}{a_0z}-1\right]\right]\Bigg) \quad \text{for} \quad z\in D^-. \qquad (33)
\end{aligned}
$$

Of particular interest is the displacement of the crack face of the elastic matrix. This may be calculated as the limit of (32) as $z \to t^-$ or by integrating

$$2\mu(u'(t)+iv'(t)) = (\kappa+1)\Phi^-(t) \quad \text{for} \quad t\in A_S$$

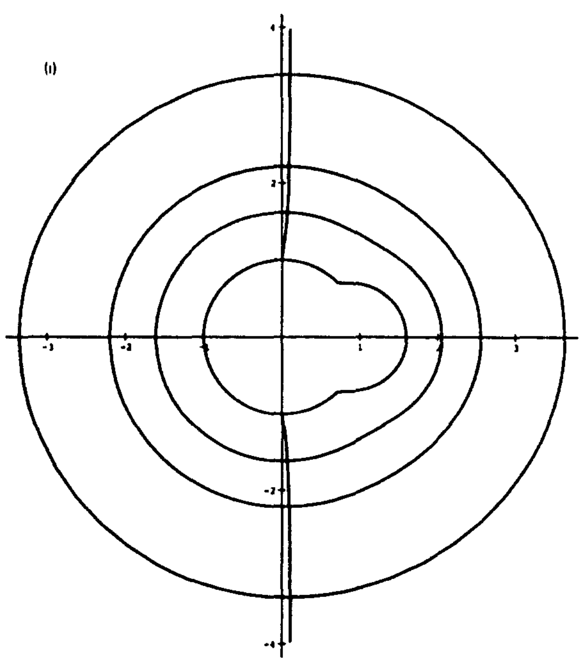

Fig. 3. The deformation of a fixed rigid inclusion bonded to an elastic matrix with a crack from $e^{-i\pi/4}$ to $e^{i\pi/4}$ for the natural loads: (i) hydrostatic tension, (ii) pure shear with $\varphi = \pi/4$, and (iii) an infinitesimal rigid rotation, $\varepsilon_\infty = 0.5$. The reference configuration is the same as in Fig. 2.

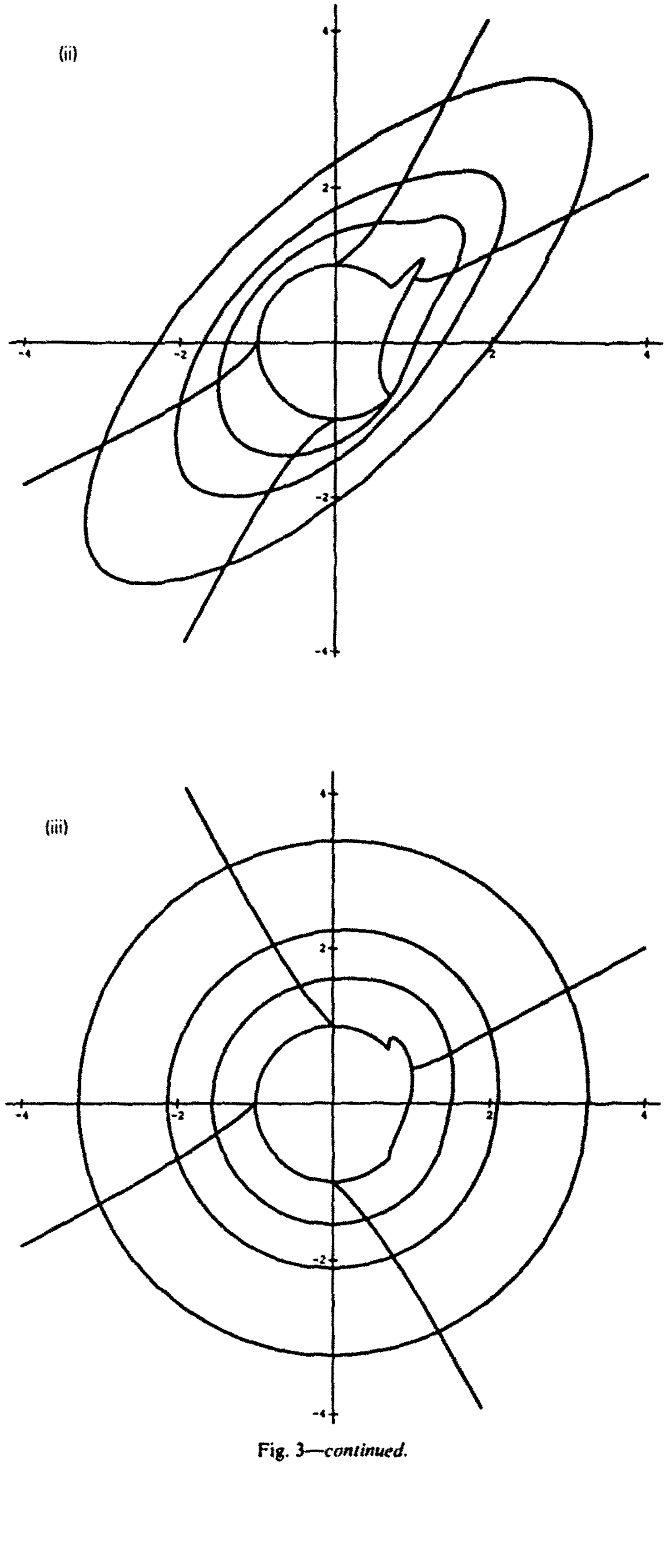

Fig. 3—*continued.*

as in Toya (1974). These displacements may be written in the simple form

$$2\mu(u(t)+iv(t)) = 2(\kappa+1)\sqrt{a_0}\left[\sin\frac{\alpha-\theta}{2}\sin\frac{\alpha+\theta}{2}\right]^{1/2}$$

$$\times\left[\tfrac{1}{4}(T_1+T_2)+\frac{2i\mu\varepsilon_\infty}{1+\kappa}-\tfrac{1}{2}(T_2-T_1)\,e^{2i\varphi}\frac{e^{-i\varphi}}{a_0}\right]e^{i(\theta/2-g(\theta))}$$

$$\text{for}\quad t = e^{i\theta}\quad\text{and}\quad g(\theta) = \frac{\ln(\kappa)}{2\pi}\ln\left[\frac{\sin\frac{\alpha+\theta}{2}}{\sin\frac{\alpha-\theta}{2}}\right]. \qquad (34)$$

From this expression, we may determine when a point on the elastic crack face will interpenetrate the inclusion. A negative radial displacement is certainly a necessary condition for interpenetration; however, it is not sufficient. In fact, if a large tangential displacement is present, a small negative radial displacement may cause the crack face to pull away from the inclusion, creating an open crack. It can be shown that if

$$u_r(t)-iu_\theta(t) = a(\theta)\,e^{ib(\theta)}\quad\text{for}\quad t = e^{i\theta}\in A_S$$

then interpenetration will occur when

$$a(\theta) < -2\cos b(\theta)\quad\text{if}\quad a(\theta)>0\quad\text{or}\quad a(\theta) > -2\cos b(\theta)\quad\text{if}\quad a(\theta)<0. \qquad (35)$$

Note that if displacements are small, i.e., $|a(\theta)| \ll 1$, then whenever the radial displacement is negative it will form a useful estimate of the maximum extent of the interpenetration region. Furthermore, this estimate is correct in the limit as $a(\theta)\to 0$. From (22) and (34), the radial and tangential displacements for each of the natural loads may be seen to be of the form

$$u_r(t)+iu_\theta(t) = k\left[\sin\frac{\alpha-\theta}{2}\sin\frac{\alpha+\theta}{2}\right]^{1/2} e^{ib(\theta)} \qquad (36)$$

where the load factor

$$k = \frac{T}{2\mu}(\kappa+1)\sqrt{a_0}\quad\text{and}\quad b(\theta) = \frac{-\theta}{2}-g(\theta)$$

for hydrostatic tension or compression,

$$k = \varepsilon_\infty\sqrt{a_0}\quad\text{and}\quad b(\theta) = \frac{\pi}{2}-\frac{\theta}{2}-g(\theta)$$

for an infinitesimally rigid rotation, and

$$k = \frac{S}{\mu\sqrt{a_0}}(\kappa+1)\quad\text{and}\quad b(\theta) = 2\varphi-\frac{3\theta}{2}-g(\theta)$$

for a pure shear. The function $g(\theta)$ is monotonically increasing and odd about $\theta = 0$ with asymptotes at $\theta = \alpha$ and $\theta = -\alpha$ (see Fig. 4). For hydrostatic tension, $b(\theta) = (-\theta/2)-g(\theta)$ will remain between $-\pi/2$ and $\pi/2$ except for very small zones near $\theta = -\alpha$ and $\theta = \alpha$. As

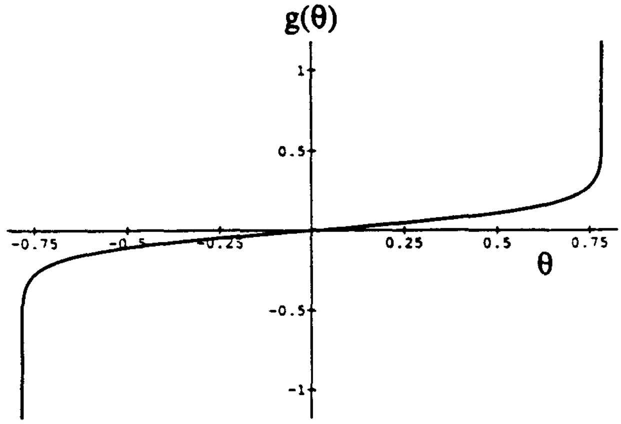

Fig. 4. The function $g(\theta)$ for $-\alpha < \theta < \alpha$, where $\alpha = \pi/4 \approx 0.785$ and $\kappa = 1.6$.

$\theta \to -\alpha$, the asymptote of $g(\theta)$ will cause $b(\theta)$ to take on values in

$$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{5\pi}{2}, \frac{7\pi}{2}\right), \ldots,$$

causing negative radial displacement for these angles. A similar behavior occurs as $\theta \to \alpha$ (see Fig. 5). This must cause interpenetration in parts of these regions since when $b(\theta) = \pi, 3\pi, \ldots$ the radial displacement is negative with zero tangential displacement. However, these zones are very small, as can be seen from Fig. 6, which illustrates how the maximum length of these zones varies as a function of crack length. Thus for hydrostatic tension, the solution predicts the formation of a large open crack symmetric about $\theta = 0$ with small zones near the tips forming alternating regions of interpenetration and open crack. For a hydrostatic compression, the previous case is reversed with a large interpenetration region formed symmetrically about $\theta = 0$ with similar small zones near the crack tips. For a positive infinitesimally rigid rotation ($\varepsilon_\infty > 0$), $b(\theta) = (\pi/2) - (\theta/2) - g(\theta)$ will now remain between $\pi/2$ and π for $-\alpha < \theta < 0$ except for the small zone near $\theta = -\alpha$ (see Fig. 5). Thus negative radial displacement is predicted throughout $-\alpha < \theta < 0$ except for a very small region near $\theta = -\alpha$. While the interpenetration condition (35) will delay interpenetration, Fig. 7 illustrates that even for relatively large values of $k = \varepsilon_\infty\sqrt{a_0}$, the region of interpenetration predicted is still quite large. Therefore the positive infinitesimally

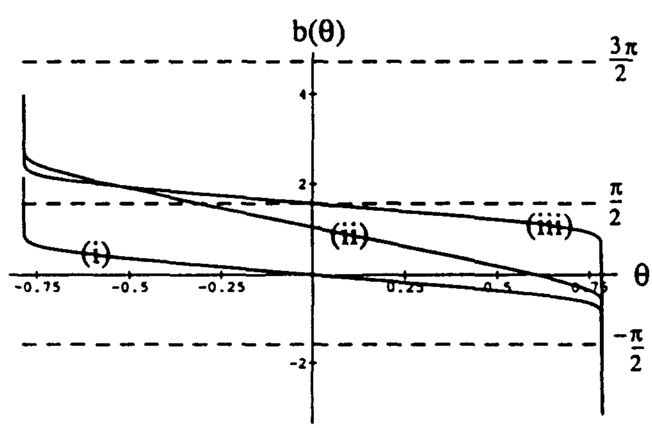

Fig. 5. The function $b(\theta)$ for $-\alpha < \theta < \alpha$, where $\alpha = \pi/4 \approx 0.785$ and $\kappa = 1.6$ for: (i) hydrostatic tension, (ii) pure shear with $\varphi = \pi/6$, and (iii) an infinitesimal rigid rotation.

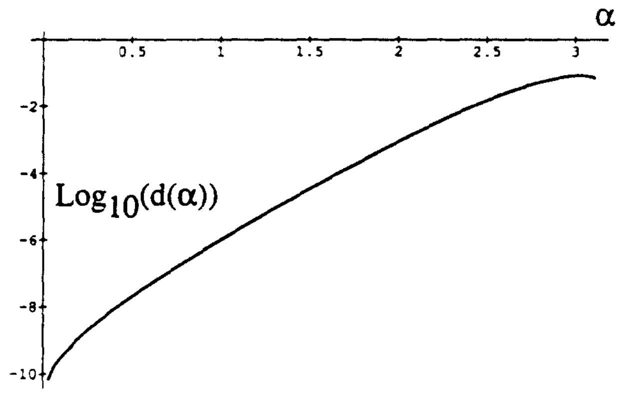

Fig. 6. $\mathrm{Log}_{10}(d(\alpha))$ plotted against α for $0 < \alpha < \pi$ where $d(\alpha)$ is the length from the crack tip to the first negative radial displacement for a crack of length 2α.

rigid rotation predicts a large region of interpenetration within $-\alpha < \theta < 0$ and open crack elsewhere except for small zones at the crack tips. Finally, for a positive pure shear ($S > 0$), we find the length of the contact region dependent on φ, the angle the positive principal stress makes with the positive real axis. The angle φ will cause

$$b(\theta) = 2\varphi - \frac{3\theta}{2} - g(\theta)$$

to translate along the y axis. Thus for a crack with $\alpha = \pi/4$ as considered in Fig. 5, if $\varphi = 0$, the maximum interpenetration region is restricted to small zones similar to the case of a hydrostatic tension. If $\varphi = \pi/4$, the maximum interpenetration region runs from $-\alpha < \theta < 0$ similar to an infinitesimally rigid rotation. If $\varphi = \pi$, the maximum interpenetration region extends almost the entire length of the crack in a manner similar to a hydrostatic compression. Figure 8 illustrates the delay of interpenetration due to (35) for large values of the load factor $k = (S/\mu\sqrt{a_0})(\kappa + 1)$ for the same situation as in Fig. 5, $\alpha = \pi/4$ and $\varphi = \pi/6$. Again, it can be seen that the region of interpenetration predicted by (35), even for large values of k, remains quite large. By constructing similar graphs as in Figs 5 and 8, interpenetration zones may be predicted for any crack of length 2α, any angle of shear φ, and any load level k.

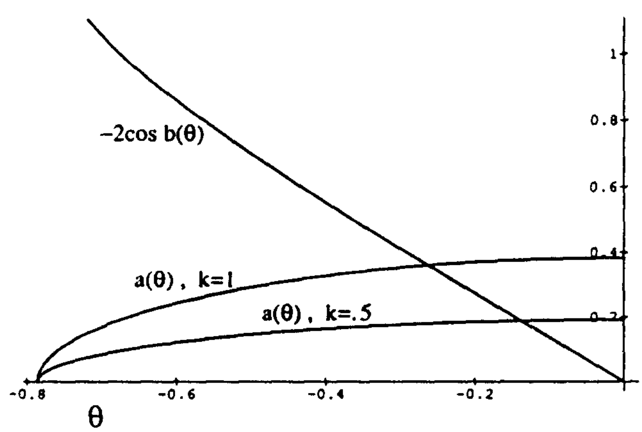

Fig. 7. The curves $a(\theta)$ for $k = 1$ and $k = 0.5$ and the curve $-2\cos b(\theta)$ for $-\alpha < \theta < 0$ for a positive infinitesimal rigid rotation. $\alpha = \pi/4$ and $\kappa = 1.6$. When $a(\theta) < -2\cos b(\theta)$ the elastic crack face will interpenetrate the inclusion. Note the maximum range of interpenetration is $(-\alpha, 0)$, the region of interpenetration for $k = 0.5$ is approximately $(-\alpha, -0.15)$, and the region of interpenetration for $k = 1$ is approximately $(-\alpha, -0.25)$.

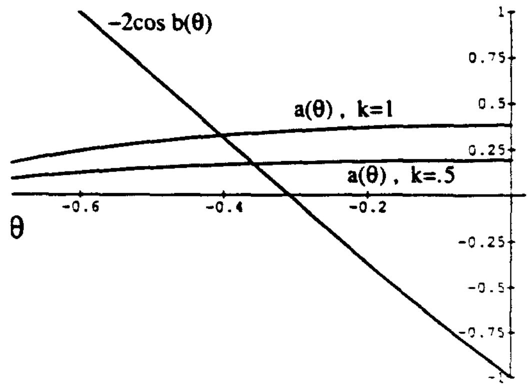

Fig. 8. The curves $a(\theta)$ for $k = 1$ and $k = 0.5$ and the curve $-2\cos b(\theta)$ for $-\alpha < \theta < 0$ for a positive pure shear. $\alpha = \pi/4$, $\kappa = 1.6$, and $\varphi = \pi/6$. Note the maximum region of interpenetration is approximately $(-\alpha, -0.3)$, the region of interpenetration for $k = 0.5$ is approximately $(-\alpha, -0.35)$, and the region of interpenetration for $k = 1$ is approximately $(-\alpha, -0.4)$.

It should be noted that while each of the natural far-field loads has an elastic crack face displacement of the form $a(\theta)\,e^{ib(\theta)}$, the superposition of these natural loads does not since

$$u_r(t) + iu_\theta(t) = a_1(\theta)\,e^{ib_1(\theta)} + a_2(\theta)\,e^{ib_2(\theta)} + a_3(\theta)\,e^{ib_3(\theta)}$$

cannot be written as $a(\theta)\,e^{ib(\theta)}$ when the functions $b_i(\theta)$ differ. To check for interpenetration in this case, one needs to determine when $|1 + u_r(t) + iu_\theta(t)| < 1$.

CONCLUSION

By considering a rigid inclusion in an elastic matrix, it has been possible in this mathematically simpler setting to analyze the behavior of an interface crack. It was seen that the problem can be decomposed into the superposition of a hydrostatic tension or compression, a pure shear, and an infinitesimally rigid rotation about infinity. For the first time, it was possible to write down a simple closed-form expression for the displacements throughout the elastic body and observe the deformation of the body. The deformation of the elastic crack face under each of the natural loads could be easily analyzed and the regions of interpenetration determined. In most instances, it was found that this region of interpenetration was not small. Only a hydrostatic tension always predicts small interpenetration zones analogous to a Griffith interface crack under a uniaxial tension normal to the crack faces. Thus, in most loading situations, if the behavior of the material near the crack is to be accurately determined, a model which allows the crack faces to come into contact is necessary. The author intends to address the contact model in future studies.

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APPENDIX

We shall first consider the function

$$\chi(z) = \left(\frac{z-e^{-i\alpha}}{z-e^{i\alpha}}\right)^{\lambda_0} \frac{1}{z-e^{-i\alpha}}$$

where

$$\lambda_0 = \frac{1}{2} - \frac{i}{2\pi}\ln(\kappa)$$

with the branch of w^{λ_0} defined by $-\alpha < \arg(w) < 2\pi-\alpha$. We shall represent the expansions of this function about zero and infinity by

$$\chi(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{for} \quad |z| < 1 \quad \text{and} \quad \chi(z) = \sum_{k=1}^{\infty} b_k z^{-k} \quad \text{for} \quad |z| > 1.$$

The first two terms of each expansion are

$$a_0 = \kappa^{(1-\alpha/\pi)}, \quad a_1 = a_0\left(\cos\alpha - \frac{\ln(\kappa)}{\pi}\sin\alpha\right), \quad b_1 = 1, \quad \text{and} \quad b_2 = \cos\alpha + \frac{\ln(\kappa)}{\pi}\sin\alpha.$$

We shall need the following two integrals:

$$\frac{1}{2\pi i}\int_{A_D} \frac{1}{\chi^+(r)}\frac{dr}{r-z} = \begin{cases} \dfrac{1}{1+\kappa}\dfrac{1}{\chi(z)} + \dfrac{1}{1+\kappa}[b_2 - z] & z \notin A_D, \\[2ex] \dfrac{1-\kappa}{2(1+\kappa)}\dfrac{1}{\chi^+(z)} + \dfrac{1}{1+\kappa}[b_2 - z] & z \in A_D \end{cases} \tag{A1}$$

$$\frac{1}{2\pi i}\int_{A_D} \frac{1}{r^2\chi^+(r)}\frac{dr}{r-z} = \begin{cases} \dfrac{1}{1+\kappa}\dfrac{1}{z^2\chi(z)} + \dfrac{1}{1+\kappa}\dfrac{1}{a_0 z^2}\left[\dfrac{a_1}{a_0}z - 1\right] & z \notin A_D, \\[2ex] \dfrac{1-\kappa}{2(1+\kappa)}\dfrac{1}{z^2\chi^+(z)} + \dfrac{1}{1+\kappa}\dfrac{1}{a_0 z^2}\left[\dfrac{a_1}{a_0}z - 1\right] & z \in A_D. \end{cases} \tag{A2}$$

Each integral may be calculated by residues using the contour shown in Fig. 9 and the fact that $\chi^+(t) = -\kappa\chi^-(t)$ for $t \in A_d$.

We also need the following results that if

$$R_1(z) = (z-e^{i\alpha})\left(\frac{z-e^{-i\alpha}}{z-e^{i\alpha}}\right)^{\lambda_0}$$

and

$$R_2(z) = (z-e^{i\alpha})\left[e^{2i\alpha}\frac{z-e^{-i\alpha}}{z-e^{i\alpha}}\right]^{\bar{\lambda}_0},$$

then

$$R_1'(z) = \left(\frac{z-e^{-i\alpha}}{z-e^{i\alpha}}\right)^{\lambda_0}\frac{1}{z-e^{-i\alpha}}\left(z-\cos\alpha - \frac{\ln(\kappa)}{\pi}\sin\alpha\right) = \chi(z)(z-b_2) \tag{A3}$$

$$R_2'(z) = \left(e^{2i\alpha}\frac{z-e^{-i\alpha}}{z-e^{i\alpha}}\right)^{\bar{\lambda}_0}\frac{1}{z-e^{-i\alpha}}\left(z-\cos\alpha + \frac{\ln(\kappa)}{\pi}\sin\alpha\right) = \frac{-e^{i\alpha}}{z}\bar{\chi}(1/z)\left(z-\frac{a_1}{a_0}\right) \tag{A4}$$

$$\left(\frac{R_1(z)}{z}\right)' = \left(\frac{z-e^{-i\alpha}}{z-e^{i\alpha}}\right)^{\lambda_0}\frac{1}{z-e^{-i\alpha}}\left[\frac{1}{z}\left(\cos\alpha - \frac{\ln(\kappa)}{\pi}\sin\alpha\right)\frac{1}{z^2}\right] = \frac{1}{z^2}\chi(z)\left(\frac{a_1}{a_0}z-1\right) \tag{A5}$$

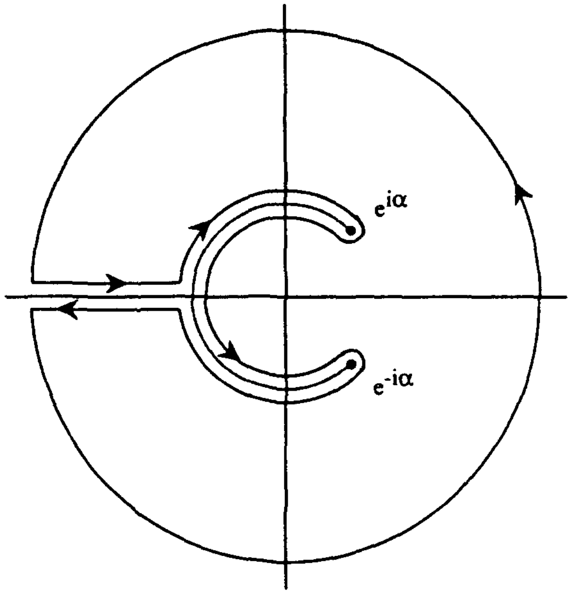

Fig. 9. Contour of integration for the integrals (A1) and (A2) in the Appendix.

$$\left(\frac{R_2(z)}{z}\right)' = \left(e^{2i\alpha}\frac{z-e^{-i\alpha}}{z-e^{i\alpha}}\right)^{\bar{\lambda}_0}\frac{1}{z-e^{-i\alpha}}\left[\frac{1}{z}\left(\cos\alpha+\frac{\ln(\kappa)}{\pi}\sin\alpha\right)-\frac{1}{z^2}\right]=\frac{e^{i\alpha}}{z^2}\bar{\chi}(1/z)\left(\frac{1}{z}-b_2\right). \tag{A6}$$

Note that it can be shown that

$$\bar{\chi}(1/z) \equiv \overline{\frac{1}{1/\bar{z}-e^{-i\alpha}}\left(\frac{1/\bar{z}-e^{-i\alpha}}{1/\bar{z}-e^{i\alpha}}\right)^{\lambda_0}} = \frac{-z\,e^{i\alpha}}{z-e^{-i\alpha}}\left(e^{2i\alpha}\frac{z-e^{-i\alpha}}{z-e^{i\alpha}}\right)^{\bar{\lambda}_0}.$$